IMO 2016 Problems with Solution Outlines

Part 1 · The paper at a glance

57th IMO | Hong Kong | July 2016

CHART **Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).**

P1	P2	P3	P4	P5	P6
Geometry	**Combinatorics**	**Number theory**	**Number theory**	**Algebra**	**Combinatorics**
G1 · Proof	C4 · Find the answer	N7 · Proof	N3 · Find the answer	A6 · Find the answer	C7 · Proof
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Geometry · G1

Proposed by Art Waeterschoot, Belgium

Triangle BCF has a right angle at B. Point A lies on line CF with F between A and C, and FA = FB. Point D satisfies DA = DC, and line AC bisects ∠DAB. Point E satisfies EA = ED, and line AD bisects ∠EAC. M is the midpoint of CF, and X is the point that makes AMXE a parallelogram (AM ∥ EX and AE ∥ MX). Prove that lines BD, FX and ME pass through one common point.

PROOF PROBLEM

KEY IDEA

The six base angles FAB, FBA, DAC, DCA, EAD and EDA are all equal. The similar isosceles triangles AFB and ADC show that B, C, D, F lie on a circle centred at M. Next, ED ∥ AC and MX = AE = MB, so X is on that circle too. A, B, M, D, E lie on a second circle, and EXMF is an isosceles trapezium, so it is cyclic. BD, FX and ME are the common chords of these three circles, so they meet at the radical centre.

Full written solutions: AoPS wiki, 2016 IMO Problem 1

2 Problem 2

Combinatorics · C4

Proposed by Trevor Tao, Australia

Find all positive integers n for which every cell of an n × n table can be filled with one of the letters I, M, O so that (1) in each row and each column, one third of the entries are I, one third are M and one third are O, and (2) on every diagonal whose number of cells is a multiple of 3, the three letters also appear equally often.

ANSWER **All n that are multiples of 9.**

Problem 2, continued

KEY IDEA

For n = 9, tile the board with 3 × 3 blocks of a single letter arranged so every row, column and diagonal balances; repeat the pattern for any multiple of 9. For the converse write n = 3k and cut the board into k^2 boxes of size 3 × 3. Add up the letter counts over every second row and second column of the boxes (rows and columns numbered 2 mod 3) and over all diagonals whose length is a multiple of 3. Each box centre is counted four times and every other cell once, so the k^2 centres alone hold equal numbers of I, M and O. Hence 3 divides k^2, and 9 divides n.

Full written solutions: AoPS wiki, 2016 IMO Problem 2

3 Problem 3

Number theory · N7

Proposed by Aleksandr Gaifullin, Russia

$P = A_1A_2...A_k$ is a convex polygon whose vertices have integer coordinates and lie on one circle. S is the area of P. An odd positive integer n is given such that the square of every side length of P is an integer divisible by n. Prove that 2S is an integer divisible by n.

PROOF PROBLEM

KEY IDEA

It is enough to treat $n = p^e$ for an odd prime p. For a triangle, Heron's formula writes $16S^2$ as a polynomial in the squared sides, which settles the case k = 3. For more sides, a Ptolemy-type relation for cyclic polygons, read through the exponent of p, shows that the square of some diagonal is also divisible by p^e. Cut along that diagonal and use induction on the number of sides.

Full written solutions: AoPS wiki, 2016 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Number theory · N3

Proposed by Gerhard Woeginger, Luxembourg

A set of positive integers is called fragrant if it has at least two elements and each element shares a prime factor with at least one other element. Let $P(n) = n^2 + n + 1$. Find the smallest positive integer b for which some non-negative integer a makes the set $\{P(a+1), P(a+2), ..., P(a+b)\}$ fragrant.

ANSWER **b = 6.**

Problem 4, continued

KEY IDEA

By the Euclidean algorithm, P(n) and P(n+1) are coprime, gcd(P(n), P(n+2)) divides 7, gcd(P(n), P(n+3)) divides 3 and gcd(P(n), P(n+4)) divides 19. With five or fewer consecutive values these three primes cannot give every element a partner. For b = 6 the Chinese remainder theorem picks a so that 19 divides P(a+1) and P(a+5), 7 divides P(a+2) and P(a+4), and 3 divides P(a+3) and P(a+6).

Full written solutions: AoPS wiki, 2016 IMO Problem 4

5 Problem 5

Algebra · A6

Proposed by Nazar Agakhanov and Ilya Bogdanov, Russia

The equation $(x - 1)(x - 2)\ldots(x - 2016) = (x - 1)(x - 2)\ldots(x - 2016)$ is written on a board, with 2016 linear factors on each side. Find the least k for which exactly k of these 4032 factors can be erased so that at least one factor stays on each side and the new equation has no real solution.

ANSWER **k = 2016.**

KEY IDEA

Each factor $x - j$ must go from at least one side, or $x = j$ is a root, so k is at least 2016. To reach 2016, keep on the left the factors with $j \equiv 0$ or $1 \pmod 4$ and on the right those with $j \equiv 2$ or $3 \pmod 4$. Group the factors in fours: $(x - 1)(x - 4)$ is always 2 less than $(x - 2)(x - 3)$, and likewise for each later group. A sign check on each interval of the real line then shows the left side is strictly smaller than the right side for every real x.

Full written solutions: AoPS wiki, 2016 IMO Problem 5

6 Problem 6

Combinatorics · C7

Proposed by Josef Tkadlec, Czech Republic

There are $n \geq 2$ line segments in the plane. Every two of them cross, and no three pass through one point. Geoff picks one endpoint of each segment and puts a frog there, facing the other endpoint. He then claps n - 1 times; at each clap every frog jumps forward to the next intersection point on its segment. Geoff wants no two frogs ever to land on the same intersection point at the same time. (a) Prove he can do this when n is odd. (b) Prove he cannot when n is even.

PROOF PROBLEM

KEY IDEA

Push the endpoints out to a large circle and label them $P_1, P_2, \ldots, P_{2n}$ in order. Since every two segments cross, each segment joins P_i to P_{i+n}. Two frogs starting at neighbouring endpoints reach their common crossing after the same number of jumps, so frogs must sit on alternate endpoints. For odd n, the endpoints $P_1, P_3, \ldots, P_{2n-1}$ use each segment once and a parity count shows no collision. For even n, alternate endpoints include both ends of one segment, which is not allowed.

Full written solutions: AoPS wiki, 2016 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2016 paper from imo-official.org. Full solutions: AoPS wiki, 2016 IMO and Evan Chen's IMO 2016 solution notes.