IMO 2017 Problems with Solution Outlines

Part 1 · The paper at a glance

58th IMO | Rio de Janeiro, Brazil | July 2017

CHART **Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).**

P1	P2	P3	P4	P5	P6
Number theory	**Algebra**	**Combinatorics**	**Geometry**	**Combinatorics**	**Number theory**
N1 · Find the answer	A6 · Find the answer	C5 · Find the answer	G2 · Proof	C4 · Proof	N7 · Proof
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Number theory · N1

Proposed by Stephan Wagner, South Africa

For each integer $a_0 > 1$, define a sequence by $a_{n+1} = \sqrt{a_n}$ if $\sqrt{a_n}$ is an integer, and $a_{n+1} = a_n + 3$ otherwise. Find all values of a_0 for which some number A appears in the sequence infinitely many times.

ANSWER **Exactly the multiples of 3.**

KEY IDEA

Work modulo 3. No perfect square leaves remainder 2, so once a term is 2 (mod 3) the sequence only adds 3 and never repeats. Being a multiple of 3 is preserved by both rules, and a multiple of 3 eventually falls into the loop 3 → 6 → 9 → 3. A start that is 1 (mod 3) is shown, by descending through smaller and smaller squares, to reach a term that is 2 (mod 3).

Full written solutions: AoPS wiki, 2017 IMO Problem 1

2 Problem 2

Algebra · A6

Proposed by Dorlir Ahmeti, Albania

Find all functions f from the real numbers to the real numbers such that $f(f(x)f(y)) + f(x + y) = f(xy)$ for all real x and y.

ANSWER **$f(x) = 0$, $f(x) = x - 1$ and $f(x) = 1 - x$.**

KEY IDEA

If $f(0) = 0$, putting $y = 0$ gives $f(x) = 0$ for all x. Otherwise f and $-f$ are solutions together, so assume $f(0) < 0$. If $f(z) = 0$ with $z \neq 1$, choosing y with $z + y = zy$ gives $f(0) = 0$, which is impossible; and $x = y = 0$ gives $f(f(0)^2) = 0$. So f vanishes only at 1, $f(0) = -1$, and $y = 1$ gives $f(x + 1) = f(x) + 1$. The hard step is injectivity: if $f(a) = f(b)$, shift a and b by a large integer and pick x, y with $x + y = a + 1$ and $xy = b$; the equation then forces $f(x)f(y) = 0$, so x or y is 1 and $a = b$. Finally $y = 0$ gives $f(-f(x)) = -1 - f(x)$; using this twice gives $f(1 + f(x)) = f(x)$, and injectivity yields $f(x) = x - 1$.

Full written solutions: AoPS wiki, 2017 IMO Problem 2

3 Problem 3

Combinatorics · C5

Proposed by Gerhard Woeginger, Austria

A hunter and an invisible rabbit start at the same point of the plane. In each round the rabbit first moves, unseen, to a point exactly 1 away. A tracking device then reports a point that is at most 1 from the rabbit's new position. Finally the hunter moves, visibly, to a point exactly 1 away. Can the hunter always choose her moves so that, whatever the rabbit does and whatever the device reports, after 10^9 rounds the distance between them is at most 100?

ANSWER **No.**

KEY IDEA

Suppose the distance is d at some moment. For the next 200 rounds the rabbit hops in a straight line along one of two directions that are mirror images in the line through the hunter and the rabbit, while the device keeps reporting points on that central line. The hunter cannot tell the two cases apart, and whichever way she goes, d^2 grows by at least 1/2. Repeating this about 2×10^4 times takes far fewer than 10^9 rounds and pushes d above 100.

Full written solutions: AoPS wiki, 2017 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Geometry · G2

Proposed by Charles Leytem, Luxembourg

R and S are different points on a circle Ω, and RS is not a diameter. Line l is tangent to Ω at R. Point T is chosen so that S is the midpoint of RT. Point J lies on the shorter arc RS of Ω, and the circumcircle Γ of triangle JST meets l at two points; A is the one closer to R. Line AJ meets Ω again at K. Prove that line KT is tangent to Γ.

PROOF PROBLEM

KEY IDEA

From the cyclic quadrilaterals, $\angle KRS = \angle KJS = \angle ATS$, so $RK \parallel AT$. The tangent at R gives $\angle ARS = \angle RKS$, so triangles ART and SKR are similar. Because $RS = ST$, the resulting ratio also makes triangles AST and TKR similar, which gives $\angle TAS = \angle KTS$. By the converse of the tangent-chord theorem, KT is tangent to Γ.

Full written solutions: AoPS wiki, 2017 IMO Problem 4

5 Problem 5

Combinatorics · C4

Proposed by Grigory Chelnokov, Russia

Let $N \geq 2$. A row of $N(N + 1)$ football players, all of different heights, stands in a line. Prove that the coach can remove $N(N - 1)$ players so that, in the remaining row of 2N players, nobody stands between the two tallest, nobody stands between the third and fourth tallest, and so on down to the two shortest.

PROOF PROBLEM

Problem 5, continued

KEY IDEA

Cut the row into N consecutive blocks of N + 1 players. Among the blocks, find the one whose second-tallest player is the tallest of all the second-tallest players. Keep the two tallest players of that block, remove the rest of the block, and remove the single tallest player from every other block. What remains is N - 1 blocks of N players, all shorter than the kept pair. Repeat.

Full written solutions: AoPS wiki, 2017 IMO Problem 5

6 Problem 6

Number theory · N7

Proposed by John Berman, USA

An ordered pair (x, y) of integers is called primitive if gcd(x, y) = 1. Given a finite set S of primitive pairs, prove that there are a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that $a_0x^n + a_1x^{n-1}y + \ldots + a_ny^n = 1$ for every (x, y) in S.

PROOF PROBLEM

KEY IDEA

Use induction on the size of S; one pair is handled by Bézout's identity. Suppose a homogeneous polynomial g equals 1 at all earlier pairs (a_i, b_i) and (c, d) is a new pair. Let h be the product of the linear forms $b_ix - a_iy$, which vanishes at every earlier pair, and let l = ux + vy with uc + vd = 1. Take $f = g^M - C \cdot l^e \cdot h$, with e chosen so that both parts have the same degree; f is still 1 at the earlier pairs and equals $g(c, d)^M - C \cdot h(c, d)$ at the new one. If h(c, d) = 0, the new pair is the negative of an earlier one, g(c, d) = ±1, and M = 2 with C = 0 works. Otherwise g(c, d) is coprime to h(c, d), so by Euler's theorem a suitable M makes $g(c, d)^M - 1$ a multiple of h(c, d), and C is the quotient.

Full written solutions: AoPS wiki, 2017 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2017 paper from imo-official.org. Full solutions: AoPS wiki, 2017 IMO and Evan Chen's IMO 2017 solution notes.