IMO 2018 Problems with Solution Outlines

Part 1 · The paper at a glance

59th IMO | Cluj-Napoca, Romania | July 2018

CHART Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).

P1	P2	P3	P4	P5	P6
Geometry	Algebra	Combinatorics	Combinatorics	Number theory	Geometry
G1 · Proof	A2 · Find the answer	C4 · Find the answer	C2 · Find the answer	N4 · Proof	G6 · Proof
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Geometry · G1

Proposed by Silouanos Brazitikos, Evangelos Psychas and Michael Sarantis, Greece

Γ is the circumcircle of an acute triangle ABC. Points D and E lie on segments AB and AC with AD = AE. The perpendicular bisectors of BD and CE meet the minor arcs AB and AC of Γ at F and G. Prove that lines DE and FG are parallel (or are the same line).

PROOF PROBLEM

KEY IDEA

Extend FD to meet Γ again at P and GE to meet Γ again at Q. Since FB = FD, an angle chase gives AP = AD; in the same way AQ = AE. So P, D, E, Q lie on a circle centred at A. Comparing inscribed angles in this circle and in Γ gives $\angle DFG = \angle PDE$, so FG ∥ DE.

Full written solutions: AoPS wiki, 2018 IMO Problem 1

2 Problem 2

Algebra · A2

Proposed by Patrik Bak, Slovakia

Find all integers $n \geq 3$ for which there are real numbers $a_1, a_2, \ldots, a_{n+2}$ with $a_{n+1} = a_1$, $a_{n+2} = a_2$ and $a_i a_{i+1} + 1 = a_{i+2}$ for $i = 1, 2, \ldots, n$.

ANSWER **All n that are multiples of 3.**

KEY IDEA

The pattern -1, -1, 2 repeated works when 3 divides n. In general, multiply the relation by a_{i+2} and also write it one step later multiplied by a_i. Summing around the cycle gives that the sum of a_i^2 equals the sum of $a_i a_{i+3}$, so the sum of $(a_i - a_{i+3})^2$ is 0 and $a_i = a_{i+3}$. If 3 does not divide n, all the numbers are equal to some a with $a^2 + 1 = a$, which has no real solution.

Full written solutions: AoPS wiki, 2018 IMO Problem 2

3 Problem 3

Combinatorics · C4

Proposed by Morteza Saghafian, Iran

An anti-Pascal triangle is a triangular array of numbers in which every number, except those in the bottom row, is the absolute value of the difference of the two numbers just below it. Is there an anti-Pascal triangle with 2018 rows that contains every integer from 1 to 1 + 2 + … + 2018 exactly once?

ANSWER **No.**

KEY IDEA

Start at the top and keep stepping down to the larger of the two numbers below. The number reached in the bottom row is a sum of 2018 different entries, so it is at least 1 + 2 + … + 2018; therefore those entries are exactly 1, 2, …, 2018 and the path ends at the largest number. Beside the path there is a smaller anti-Pascal triangle with at least 1008 rows that avoids the numbers 1 to 2018. Running the same argument inside it produces a number larger than the maximum.

Full written solutions: AoPS wiki, 2018 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Combinatorics · C2

Proposed by Gurgen Asatryan, Armenia

A site is a point (x, y) where x and y are positive integers at most 20. All 400 sites start empty. Amy and Ben take turns placing stones, Amy first. Amy places a red stone on an empty site so that no two red stones are at distance $\sqrt{5}$. Ben places a blue stone on any empty site. They stop when a player cannot move. Find the largest K such that Amy can always place at least K red stones, however Ben plays.

ANSWER **$K = 100$.**

KEY IDEA

Two sites at distance $\sqrt{5}$ are a knight's move apart, so they have opposite chessboard colours. If Amy plays only on the 200 sites of one colour she never breaks the rule and gets at least 100 of them. For Ben, cut the board into 25 blocks of size 4×4; each block splits into four closed knight cycles of length 4. Whenever Amy plays in a cycle, Ben takes the opposite corner of that cycle, which blocks the other two sites for Amy. She gets at most one stone per cycle, and there are 100 cycles.

Full written solutions: AoPS wiki, 2018 IMO Problem 4

5 Problem 5

Number theory · N4

Proposed by Bayarmagnai Gombodorj, Mongolia

$a_1, a_2, \ldots$ is an infinite sequence of positive integers. There is an integer $N > 1$ such that, for every $n \geq N$, the number $a_1/a_2 + a_2/a_3 + \ldots + a_{n-1}/a_n + a_n/a_1$ is an integer. Prove that there is a positive integer M with $a_m = a_{m+1}$ for all $m \geq M$.

PROOF PROBLEM

KEY IDEA

Subtracting the sums for n and $n + 1$ shows that $a_n/a_{n+1} + (a_{n+1} - a_n)/a_1$ is an integer for all $n \geq N$. Fix a prime p and compare exponents of p: if the exponent in a_{n+1} is larger than in a_n, it must equal the exponent in a_1; if it is smaller, it is still at least the exponent in a_1. So for a prime not dividing a_1 the exponent never increases, and for a prime dividing a_1 it either never increases or jumps to the exponent in a_1 and then stays there. Once the finitely many primes dividing a_1 have settled, a_{n+1} divides a_n for every n, so the sequence is eventually constant.

Full written solutions: AoPS wiki, 2018 IMO Problem 5

6 Problem 6

Geometry · G6

Proposed by Tomasz Ciesla, Poland

$ABCD$ is a convex quadrilateral with $AB \cdot CD = BC \cdot DA$. Point X lies inside $ABCD$ so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle BXA + \angle DXC = 180°$.

PROOF PROBLEM

KEY IDEA

Invert the figure about X, and write A', B', C', D' for the images. The condition $AB \cdot CD = BC \cdot DA$ is kept: $A'B' \cdot C'D' = B'C' \cdot D'A'$. The two angle conditions show that the angles of the new quadrilateral at A', B', C', D' equal the angles of $ABCD$ at B, C, D, A. A convex quadrilateral with $AB \cdot CD = BC \cdot DA$ is determined up to similarity by its angles, so $D'A'B'C'$ is similar to $ABCD$. Carrying X across by this similarity gives a point inside $ABCD$ that is the isogonal conjugate of X. A point inside a convex quadrilateral has an isogonal conjugate exactly when $\angle BXA + \angle DXC = 180°$.

Full written solutions: AoPS wiki, 2018 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2018 paper from imo-official.org. Full solutions: AoPS wiki, 2018 IMO and Evan Chen's IMO 2018 solution notes.