IMO 2019 Problems with Solution Outlines

Part 1 · The paper at a glance

60th IMO | Bath, United Kingdom | July 2019

CHART **Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).**

P1	P2	P3	P4	P5	P6
Algebra	**Geometry**	**Combinatorics**	**Number theory**	**Combinatorics**	**Geometry**
A1 · Find the answer	G3 · Proof	C5 · Proof	N1 · Find the answer	C3 · Find the answer	G7 · Proof
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Algebra · A1

Proposed by Liam Baker, South Africa

Find all functions f from the integers to the integers such that $f(2a) + 2f(b) = f(f(a + b))$ for all integers a and b.

ANSWER **$f(n) = 0$ for all n, or $f(n) = 2n + c$ for any fixed integer c.**

KEY IDEA

Put $a = 0$ to get $f(f(b)) = 2f(b) + f(0)$. Put $b = 0$ and compare to get $f(2a) = 2f(a) - f(0)$. Substituting both back gives $f(a) + f(b) = f(a + b) + f(0)$, so $f(n) - f(0)$ is additive on the integers and $f(n) = kn + c$. Putting this into the original equation leaves $k = 0$ with $c = 0$, or $k = 2$ with any c.

Full written solutions: AoPS wiki, 2019 IMO Problem 1

2 Problem 2

Geometry · G3

Proposed by Anton Trygub, Ukraine

In triangle ABC, A_1 lies on side BC and B_1 lies on side AC. Points P and Q lie on segments AA_1 and BB_1 with $PQ \parallel AB$. P_1 is on line PB_1, with B_1 strictly between P and P_1, and $\angle PP_1C = \angle BAC$. Q_1 is on line QA_1, with A_1 strictly between Q and Q_1, and $\angle CQ_1Q = \angle CBA$. Prove that P, Q, P_1, Q_1 lie on one circle.

PROOF PROBLEM

KEY IDEA

Let AA_1 and BB_1 meet the circumcircle of ABC again at A_2 and B_2. Because $PQ \parallel AB$, the points P, Q, A_2, B_2 are concyclic. The angle condition gives $\angle PP_1C = \angle BAC = \angle BB_2C$, so C, B_1, P_1, B_2 are concyclic; likewise C, A_1, Q_1, A_2. A further angle chase places P_1 and Q_1 on the circle through P, Q, A_2, B_2.

Full written solutions: AoPS wiki, 2019 IMO Problem 2

3 Problem 3

Combinatorics · C5

Proposed by Adrian Beker, Croatia

A social network has 2019 users; friendship is mutual. An event of this kind may happen repeatedly: three users A, B, C, where A is friends with both B and C but B and C are not friends, change so that B and C become friends and A stops being friends with both. At the start 1010 users have 1009 friends each and 1009 users have 1010 friends each. Prove that some sequence of events leaves every user with at most one friend.

PROOF PROBLEM

KEY IDEA

Model the network as a graph. The starting graph is connected, is not complete, and has a vertex of odd degree. Each event removes one edge in total and keeps the parity of every degree. The proof shows that, as long as some component has at least three vertices, an event can be chosen so that every such component stays non-complete and keeps an odd-degree vertex. The number of edges keeps falling, so the process ends with components of one or two vertices.

Full written solutions: AoPS wiki, 2019 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Number theory · N1

Proposed by Gabriel Chicas Reyes, El Salvador

Find all pairs (k, n) of positive integers such that $k! = (2^n - 1)(2^n - 2)(2^n - 4)\ldots(2^n - 2^{n-1})$.

ANSWER **$(k, n) = (1, 1)$ and $(3, 2)$.**

KEY IDEA

The power of 2 dividing the right side is 2 to the power $n(n-1)/2$. The power of 2 dividing $k!$ is less than 2^k, so $k > n(n-1)/2$. The right side is smaller than 2 to the power n^2. For $n \geq 6$ the factorial of $n(n-1)/2$ already exceeds that bound, so only $n \leq 5$ remains, and a direct check leaves $n = 1$ and $n = 2$.

Full written solutions: AoPS wiki, 2019 IMO Problem 4

5 Problem 5

Combinatorics · C3

Proposed by David Altizio, USA

A row of n coins each shows H or T. Repeat this step: if exactly $k > 0$ coins show H, turn over the k-th coin from the left; if no coin shows H, stop. (a) Prove that the process stops after finitely many steps for every starting row. (b) Let $L(C)$ be the number of steps from starting row C. Find the average of $L(C)$ over all 2^n starting rows.

ANSWER **The average is $n(n+1)/4$.**

Problem 5, continued

KEY IDEA

Use induction on n by looking at the last coin. If the last coin shows T it is never turned, so the game is the (n - 1)-coin game on the other coins. If it shows H, a matching argument shows the game takes exactly n more steps than a corresponding (n - 1)-coin game. Both parts follow: the process stops, and the average E(n) satisfies $E(n) = E(n - 1) + n/2$.

Full written solutions: AoPS wiki, 2019 IMO Problem 5

6 Problem 6

Geometry · G7

Proposed by Anant Mudgal, India

I is the incentre of an acute triangle ABC with AB ≠ AC. The incircle ω touches BC, CA, AB at D, E, F. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at Q. Prove that lines DI and PQ meet on the line through A perpendicular to AI.

PROOF PROBLEM

KEY IDEA

Let T be the point where DI meets the line through A perpendicular to AI; we must show P, Q, T are collinear. Invert in the incircle. A, B, C go to the midpoints A′, B′, C′ of EF, FD, DE; T goes to the foot T′ of the perpendicular from A′ to line DI; and the two circles become the circles through P, E, C′ and through P, F, B′, meeting again at Q′. The claim becomes: P, I, Q′, T′ lie on one circle. An angle chase shows Q′ lies on line B′C′. If G is the point of the incircle opposite D, then P, A′, G are collinear, because PERF is a harmonic quadrilateral and GR ∥ EF. Let M be the midpoint of DA′; P and T′ lie on the circle with diameter DA′, centred at M, and two more angle chases show that T′ and Q′ both lie on the circle through P, M, I.

Full written solutions: AoPS wiki, 2019 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2019 paper from imo-official.org. Full solutions: AoPS wiki, 2019 IMO and Evan Chen's IMO 2019 solution notes.