IMO 2020 Problems with Solution Outlines

Part 1 · The paper at a glance

61st IMO | St Petersburg, Russia (held remotely) | September 2020

CHART Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).

P1	P2	P3	P4	P5	P6
Geometry	Algebra	Combinatorics	Combinatorics	Number theory	Geometry
G2 · Proof	A4 · Proof	C6 · Proof	C3 · Find the answer	N3 · Find the answer	G9 · Proof
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Geometry · G2

Proposed by Dominik Burek, Poland

P is a point inside a convex quadrilateral ABCD such that $\angle PAD : \angle PBA : \angle DPA = 1 : 2 : 3 = \angle CBP : \angle BAP : \angle BPC$. Prove that the internal bisectors of $\angle ADP$ and $\angle PCB$ and the perpendicular bisector of AB meet at one point.

PROOF PROBLEM

KEY IDEA

Let O be the circumcentre of triangle PAB; it lies on the perpendicular bisector of AB. Write $\angle PAD = \varphi$, so $\angle PBA = 2\varphi$ and $\angle DPA = 3\varphi$. Then $\angle ADP = 180° - 4\varphi$ and the central angle $\angle AOP = 2\angle ABP = 4\varphi$, so A, D, P, O are concyclic. Since OA = OP, the point O is the midpoint of arc AP, which means DO bisects $\angle ADP$. The same argument on the other side shows CO bisects $\angle PCB$.

Full written solutions: AoPS wiki, 2020 IMO Problem 1

2 Problem 2

Algebra · A4

Proposed by Stijn Cambie, Belgium

Real numbers $a \geq b \geq c \geq d > 0$ satisfy $a + b + c + d = 1$. Prove that $(a + 2b + 3c + 4d) \cdot a^a b^b c^c d^d < 1$.

PROOF PROBLEM

KEY IDEA

Weighted AM-GM with weights a, b, c, d gives $a^a b^b c^c d^d \leq a^2 + b^2 + c^2 + d^2$. It then suffices to show $(a + 2b + 3c + 4d)(a^2 + b^2 + c^2 + d^2) < (a + b + c + d)^3$. Expanding the cube and using $a \geq b \geq c \geq d$, every term on the left is matched by terms on the right with something positive to spare.

Full written solutions: AoPS wiki, 2020 IMO Problem 2

3 Problem 3

Combinatorics · C6

Proposed by Milan Haiman (Hungary) and Carl Schildkraut (USA)

There are 4n pebbles of weights 1, 2, 3, ..., 4n. Each pebble is painted in one of n colours and there are four pebbles of each colour. Prove that the pebbles can be split into two piles of equal total weight so that each pile has two pebbles of each colour.

PROOF PROBLEM

KEY IDEA

Pair the pebble of weight k with the pebble of weight 4n + 1 - k; every pair weighs 4n + 1. Draw a graph whose vertices are the colours and whose edges are the 2n pairs, each joining the colours of its two pebbles. Every vertex has degree 4, so each component has an Euler circuit with an even number of edges. Colour the edges alternately along each circuit: each colour then has two edges of each kind, which gives the two piles.

Full written solutions: AoPS wiki, 2020 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Combinatorics · C3

Proposed by Tejaswi Navilarekallu, India

Let $n > 1$. On a mountain slope there are n^2 stations at different heights. Two companies, A and B, each run k cable cars; each car goes from one station to a higher one. For each company the k cars have k different starting stations and k different finishing stations, and a car that starts higher also finishes higher. Two stations are linked by a company if you can travel from the lower to the higher using one or more of its cars. Find the smallest k that guarantees two stations linked by both companies.

ANSWER $k = n^2 - n + 1$.

KEY IDEA

The cars of one company join the stations into chains; with k cars there are $n^2 - k$ chains. If $k = n^2 - n + 1$ each company has $n - 1$ chains, so there are at most $(n - 1)^2$ pairs (chain of A, chain of B). Since $(n - 1)^2 < n^2$, two stations share both chains. For $k = n^2 - n$, number the stations by height, let A join i to i + 1 except when i is a multiple of n, and let B join i to i + n; no two stations are linked by both.

Full written solutions: AoPS wiki, 2020 IMO Problem 4

5 Problem 5

Number theory · N3

Proposed by Oleg Košik, Estonia

A deck of $n > 1$ cards has a positive integer on each card. For every pair of cards, the arithmetic mean of their two numbers equals the geometric mean of the numbers on some collection of one or more cards. For which n does it follow that all the numbers are equal?

Problem 5, continued

ANSWER **For every $n > 1$.**

KEY IDEA

Divide all the numbers by their greatest common divisor, so no prime divides every card. Suppose the cards are not all equal; then the largest card M is greater than 1. Pick a prime p dividing M and let a be the largest card not divisible by p. The mean of a and M is larger than a and is not divisible by p (for $p = 2$ it is not even an integer, which a geometric mean of integers cannot be unless it is whole). But a geometric mean that is not divisible by p can only use cards not divisible by p, and those are all at most a. This is a contradiction.

Full written solutions: AoPS wiki, 2020 IMO Problem 5

6 Problem 6

Geometry · G9

Proposed by Ting-Feng Lin and Hung-Hsun Hans Yu, Taiwan

Prove that there is a positive constant c such that the following holds. Let $n > 1$ and let S be a set of n points in the plane in which every two points are at least 1 apart. Then some line separates S (points of S lie on both sides) and every point of S is at distance at least $c \cdot n^{-1/3}$ from the line.

PROOF PROBLEM

KEY IDEA

Let δ be the largest gap between neighbouring projections of S onto any line; a line through the middle of that gap separates S and is at distance $\delta/2$ from every point, so it is enough to show $\delta \geq c \cdot n^{-1/3}$. Take the two points A, B of S that are farthest apart and let $R = AB$. Projecting onto AB gives $R \leq (n-1)\delta$. All of S lies in the disc with centre B and radius R, so the points whose projection onto AB is within 1/2 of A lie in a thin cap of that disc, cut off by a chord XY perpendicular to AB of length less than $2\sqrt{R} < 2\sqrt{(n\delta)}$. Projections onto AB are at most δ apart, so the cap holds more than $1/(2\delta)$ points. Two points of the cap differ by at most 1/2 in the AB direction but are at least 1 apart, so their projections onto XY are at least $\sqrt{3}/2$ apart, and $XY \geq (\sqrt{3}/2)(1/(2\delta) - 1)$. Comparing the two bounds on XY gives $\delta \geq c \cdot n^{-1/3}$.

Full written solutions: AoPS wiki, 2020 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2020 paper from imo-official.org. Full solutions: AoPS wiki, 2020 IMO and Evan Chen's IMO 2020 solution notes.