IMO 2021 Problems with Solution Outlines

Part 1 · The paper at a glance

62nd IMO | St Petersburg, Russia (held remotely) | July 2021

CHART **Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).**

P1	P2	P3	P4	P5	P6
Number theory	**Algebra**	**Geometry**	**Geometry**	**Combinatorics**	**Algebra**
N2 · Proof	A4 · Proof	G7 · Proof	G2 · Proof	C3 · Proof	A6 · Proof
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Number theory · N2

Proposed by Australia

Let $n \geq 100$. The numbers $n, n+1, \ldots, 2n$ are written on separate cards, which are shuffled and split into two piles. Prove that at least one pile contains two cards whose numbers add up to a perfect square.

PROOF PROBLEM

KEY IDEA

It is enough to find three cards $a < b < c$ whose three pairwise sums are all squares, because two of them must share a pile. Ask for $a + b = (2k-1)^2$, $a + c = (2k)^2$ and $b + c = (2k+1)^2$. This gives $a = 2k^2 - 4k$, $b = 2k^2 + 1$ and $c = 2k^2 + 4k$. For every $n \geq 100$ some k has $n \leq 2k^2 - 4k$ and $2k^2 + 4k \leq 2n$.

Full written solutions: AoPS wiki, 2021 IMO Problem 1

2 Problem 2

Algebra · A4

Proposed by Calvin Deng, Canada

Prove that for all real numbers $x_1, \ldots, x_n$, the sum of $\sqrt{|x_i - x_j|}$ over all i and j is at most the sum of $\sqrt{|x_i + x_j|}$ over all i and j.

PROOF PROBLEM

KEY IDEA

Adding the same number t to every x_i leaves the left side unchanged. Study the right side as a function of t: it is concave between the points where some $x_i + x_j + 2t$ is zero, so its minimum happens when some $x_i = 0$ or some $x_i = -x_j$. In either case those terms can be removed without changing the difference of the two sides, and induction on n finishes. A second proof writes $\sqrt{|a|}$ as an integral and turns the difference into the integral of a square.

Full written solutions: AoPS wiki, 2021 IMO Problem 2

3 Problem 3

Geometry · G7

Proposed by Mykhailo Shtandenko, Ukraine

D is a point inside an acute triangle ABC with AB > AC and ∠DAB = ∠CAD. Point E on segment AC satisfies ∠ADE = ∠BCD, and point F on segment AB satisfies ∠FDA = ∠DBC. Point X on line AC satisfies CX = BX. O_1 and O_2 are the circumcentres of triangles ADC and EXD. Prove that lines BC, EF and O_1O_2 meet at one point.

PROOF PROBLEM

KEY IDEA

Let Q be the isogonal conjugate of D; it lies on line AD, and the angle conditions make C, E, D, Q concyclic and B, F, D, Q concyclic. So AE · AC = AD · AQ = AF · AB and B, C, E, F are concyclic. Let T be the point where EF meets BC. An angle count shows that the circles BCD and DEF touch at D, so their common tangent passes through T and TD^2 = TB · TC = TE · TF. Let M be the Miquel point of BCEF; it lies on line AT and TM · TA = TD^2. One shows B, X, M, E are concyclic, and then that the circles ACD, EXD and BDM all pass through D and one more common point, so their three centres are collinear. Inversion in the circle with centre T and radius TD swaps A with M and C with B and fixes D, so it swaps the circles ACD and BDM; hence T lies on the line through their centres, which is O_1O_2.

Full written solutions: AoPS wiki, 2021 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Geometry · G2

Proposed by Dominik Burek and Tomasz Ciesla, Poland

Γ is a circle with centre I, and ABCD is a convex quadrilateral whose four sides are tangent to Γ. Ω is the circumcircle of triangle AIC. The extension of BA beyond A meets Ω at X and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T. Prove that AD + DT + TX + XA = CD + DY + YZ + ZC.

PROOF PROBLEM

KEY IDEA

An angle chase in Ω, using that IA bisects ∠DAB and IC bisects ∠BCD, gives IX = IY and IT = IZ. So X and Y, and also T and Z, are mirror images in the diameter of Ω through I, which gives TX = YZ. Points at the same distance from I also have equal tangent lengths to Γ. Writing XA, ZC, DT, DY, AD and CD as sums and differences of tangent lengths makes the remaining parts of the two sides equal.

Full written solutions: AoPS wiki, 2021 IMO Problem 4

5 Problem 5

Combinatorics · C3

Proposed by Spain

Two squirrels have 2021 walnuts numbered 1 to 2021, placed around a tree in a circle in some order. They make 2021 moves; in move k the two walnuts next to walnut k swap places. Prove that for some k the two walnuts swapped in move k have numbers a and b with $a < k < b$.

PROOF PROBLEM

KEY IDEA

Suppose it never happens. Call walnut j touched once move j has been made. In move k, the two neighbours of walnut k are then both touched or both untouched, and swapping two walnuts of the same kind changes nothing. Count the adjacent pairs of touched walnuts: the count starts at 0 and each move changes it by 0 or 2. At the end all 2021 adjacent pairs are touched, an odd number. This is a contradiction.

Full written solutions: AoPS wiki, 2021 IMO Problem 5

6 Problem 6

Algebra · A6

Proposed by Austria

Let $m \geq 2$ be an integer, A a finite set of integers, and $B_1, B_2, \ldots, B_m$ subsets of A. For each $k = 1, 2, \ldots, m$ the sum of the elements of B_k is m^k. Prove that A has at least $m/2$ elements.

PROOF PROBLEM

KEY IDEA

Let A have k elements. Consider the m^m numbers $c_1 m + c_2 m^2 + \ldots + c_m m^m$ with each c_j between 0 and $m - 1$; they are all different. Each is a combination of the elements of A in which every coefficient lies between 0 and $m(m - 1)$. There are at most $(m^2 - m + 1)^k$ such combinations, so $(m^2 - m + 1)^k \geq m^m$. Since $m^2 - m + 1 < m^2$, this gives $k > m/2$.

Full written solutions: AoPS wiki, 2021 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2021 paper from imo-official.org. Full solutions: AoPS wiki, 2021 IMO and Evan Chen's IMO 2021 solution notes.