IMO 2022 Problems with Solution Outlines

Part 1 · The paper at a glance

63rd IMO | Oslo, Norway | July 2022

CHART Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).

P1	P2	P3	P4	P5	P6
Combinatorics	Algebra	Number theory	Geometry	Number theory	Combinatorics
C2 · Find the answer	A3 · Find the answer	N7 · Proof	G1 · Proof	N4 · Find the answer	C8 · Find the answer
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Combinatorics · C2

Proposed by Baptiste Serraille, France

A bank issues aluminium coins (A) and bronze coins (B). Marianne has n coins of each type in a row, in some order. A chain is a maximal block of consecutive coins of the same type. For a fixed k with $1 \le k \le 2n$ she repeats this step: find the chain containing the k-th coin from the left and move the whole chain to the left end of the row. Find all pairs (n, k) such that, for every starting order, at some moment the leftmost n coins are all of the same type.

ANSWER **All pairs with $n \le k \le \lceil 3n/2 \rceil$.**

KEY IDEA

If $k < n$, the row A…A B…B A with $n - 1$ coins in the first block never changes. If $k > \lceil 3n/2 \rceil$, a row of four chains with lengths $\lfloor n/2 \rfloor$, $\lfloor n/2 \rfloor$, $\lceil n/2 \rceil$, $\lceil n/2 \rceil$ just rotates for ever. Now let $n \le k \le \lceil 3n/2 \rceil$. The number of chains never goes up. When $k \ge n$ and the first n coins are not yet all the same, it stays the same only if the moved chain is the last one, has a different type from the first chain, and is long enough to reach back to the k-th coin, that is, has length at least $2n + 1 - k$. If this happened three times in a row with more than two chains, the chains moved first and third would be of the same type, so $2(2n + 1 - k) \le n$, giving $k \ge 3n/2 + 1$, which is impossible. So the number of chains drops to two, and then the first n coins match.

Full written solutions: AoPS wiki, 2022 IMO Problem 1

2 Problem 2

Algebra · A3

Proposed by Merlijn Staps, Netherlands

Find all functions f from the positive real numbers to the positive real numbers such that for every $x > 0$ there is exactly one $y > 0$ with $x \cdot f(y) + y \cdot f(x) \le 2$.

ANSWER **$f(x) = 1/x$.**

Problem 2, continued

KEY IDEA

Call (x, y) a good pair if $x \cdot f(y) + y \cdot f(x) \leq 2$. If the partner y of x were different from x, then (x, x) and (y, y) are not good, so $f(x) > 1/x$ and $f(y) > 1/y$. That gives $x \cdot f(y) + y \cdot f(x) > x/y + y/x \geq 2$, a contradiction. So every x is its own partner and $f(x) \leq 1/x$. If $f(x)$ were strictly less than $1/x$, a number y close to x would also be good with x, so $f(x) = 1/x$.

Full written solutions: AoPS wiki, 2022 IMO Problem 2

3 Problem 3

Number theory · N7

Proposed by Ankan Bhattacharya, USA

Let k be a positive integer and S a finite set of odd primes. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around a circle so that the product of any two neighbours has the form $x^2 + x + k$ for some positive integer x.

PROOF PROBLEM

KEY IDEA

Look at the largest prime p in S. A neighbour q of p needs $x^2 + x + k$ to be divisible by p, and a quadratic has at most two roots modulo p. Size bounds then show that p has at most two possible neighbours in S, so its neighbours are forced. One checks that these two neighbours could also sit next to each other, so removing p leaves a valid circle for a smaller set, and induction finishes.

Full written solutions: AoPS wiki, 2022 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Geometry · G1

Proposed by Patrik Bak, Slovakia

ABCDE is a convex pentagon with BC = DE. A point T inside it satisfies TB = TD, TC = TE and ∠ABT = ∠TEA. Line AB meets lines CD and CT at P and Q, with P, B, A, Q in this order on the line. Line AE meets lines CD and DT at R and S, with R, E, A, S in this order. Prove that P, S, Q, R lie on one circle.

PROOF PROBLEM

KEY IDEA

Triangles TBC and TDE are congruent (three equal sides), so ∠BTC = ∠DTE and hence ∠QTB = ∠STE. With ∠ABT = ∠TEA, triangles TBQ and TES are similar. Using TB = TD and TE = TC this gives $TD \cdot TS = TC \cdot TQ$, so C, D, Q, S are concyclic. A short angle chase with this circle and the similar triangles shows ∠RPQ = ∠RSQ, which puts P, S, Q, R on one circle.

Full written solutions: AoPS wiki, 2022 IMO Problem 4

5 Problem 5

Number theory · N4

Proposed by Tijs Buggenhout, Belgium

Find all triples (a, b, p) of positive integers with p prime and $a^p = b! + p$.

ANSWER **$(a, b, p) = (2, 2, 2)$ and $(3, 4, 3)$.**

KEY IDEA

Compare b with p. If $b < p$, then a must exceed b and the two sides cannot match. If $b \geq 2p$, then p divides a, so p^p divides a^p, while $b! + p$ is divisible by p only once. So $p \leq b < 2p$ and p divides a; checking the other prime factors of a and its size then forces $a = p$. The equation $p^p - p = b!$ is finally settled by counting powers of 2 and comparing sizes, which leaves $p = 2$ and $p = 3$.

Full written solutions: AoPS wiki, 2022 IMO Problem 5

6 Problem 6

Combinatorics · C8

Proposed by Nikola Petrović, Serbia

An $n \times n$ board contains the numbers 1 to n^2, one per cell. A valley is a cell whose number is smaller than the numbers in all neighbouring cells (cells sharing a side). An uphill path is a sequence of one or more cells that starts at a valley, moves between neighbouring cells, and has increasing numbers. Find, in terms of n, the smallest possible total number of uphill paths.

ANSWER **$2n^2 - 2n + 1$.**

KEY IDEA

For the bound, every pair of neighbouring cells is the last step of at least one uphill path: walk down from the smaller cell until a valley is reached. There are $2n(n - 1)$ neighbouring pairs, and the cell with number 1 is a valley that counts as a path by itself. For the construction, choose a set T of cells that forms a tree (connected, no cycles) such that no two cells outside T are neighbours. Put 1 on a cell of T, then keep giving the next number to a cell of T next to one already numbered, and finally put the largest numbers on the cells outside T. Then 1 is the only valley and every neighbouring pair is the last step of exactly one uphill path.

Full written solutions: AoPS wiki, 2022 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2022 paper from imo-official.org. Full solutions: AoPS wiki, 2022 IMO and Evan Chen's IMO 2022 solution notes.