IMO 2023 Problems with Solution Outlines

Part 1 · The paper at a glance

64th IMO | Chiba, Japan | July 2023

CHART Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).

P1	P2	P3	P4	P5	P6
Number theory	**Geometry**	**Algebra**	**Algebra**	**Combinatorics**	**Geometry**
N1 · Find the answer	G4 · Proof	A6 · Find the answer	A3 · Proof	C3 · Find the answer	G8 · Proof
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Number theory · N1

Proposed by Santiago Rodriguez, Colombia

Find all composite integers $n > 1$ with this property: if $1 = d_1 < d_2 < \ldots < d_k = n$ are all the positive divisors of n, then d_i divides $d_{i+1} + d_{i+2}$ for every i from 1 to $k - 2$.

ANSWER $n = p^r$ for a prime p and an integer $r \geq 2$.

KEY IDEA

Prime powers work because each divisor divides all larger ones. For the converse, start at the top: d_{k-2} divides $d_{k-1} + n$ and divides n, so it divides d_{k-1}. Since $d_{k-1} = n/d_2$ and $d_{k-2} = n/d_3$, this says d_2 divides d_3. Let $p = d_2$. If p divides d_i and d_{i+1}, then p divides $d_{i+1} + d_{i+2}$ (a multiple of d_i), so p divides d_{i+2}. Hence p divides every divisor greater than 1, and n is a power of p.

Full written solutions: AoPS wiki, 2023 IMO Problem 1

2 Problem 2

Geometry · G4

Proposed by Tiago Mourão and Nuno Arala, Portugal

ABC is an acute triangle with $AB < AC$ and circumcircle Ω. S is the midpoint of the arc CB of Ω that contains A. The perpendicular from A to BC meets BS at D and meets Ω again at E. The line through D parallel to BC meets line BE at L. The circumcircle ω of triangle BDL meets Ω again at P. Prove that the tangent to ω at P meets line BS at a point on the internal bisector of $\angle BAC$.

PROOF PROBLEM

Problem 2, continued

KEY IDEA

Let M be the midpoint of the arc BC not containing A, so AM is the internal bisector of ∠BAC and SM is a diameter of Ω; let F be the point of Ω opposite A. Angle chasing gives two collinearities: L, P, S are on one line, and P, D, F are on one line. Let X be the point where AM meets BS and Y the point where line PF meets AM. Since AE ∥ SM and AM ∥ SF (AMFS is a rectangle), the similar triangles AXD, MXS and XDY, SDF give AX = XY; as ∠APY = 90°, X is the midpoint of the hypotenuse of triangle APY and XP = XA. From XP = XA, comparing arcs of Ω shows that the angle between XP and PL equals the inscribed angle PDL of ω, so XP is tangent to ω. Thus the tangent at P passes through X, which lies on BS and on the bisector.

Full written solutions: AoPS wiki, 2023 IMO Problem 2

3 Problem 3

Algebra · A6

Proposed by Ivan Chan, Malaysia

For each integer $k \geq 2$, find all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there is a polynomial $P(x) = x^k + c_{k-1}x^{k-1} + \ldots + c_1x + c_0$, with non-negative integer coefficients, such that $P(a_n) = a_{n+1}a_{n+2}\ldots a_{n+k}$ for every $n \geq 1$.

ANSWER **Exactly the arithmetic progressions $a_n = a_1 + (n-1)d$ with integer $d \geq 0$.**

KEY IDEA

First show the sequence never decreases, by comparing the products for n and n + 1. Next show the gaps $a_{n+j} - a_n$ stay bounded, because $P(a_n)$ is close to a_n^k. A bounded set of gap patterns must repeat, and matching the product with the polynomial then forces a constant difference d, with $P(x) = (x + d)(x + 2d)\ldots(x + kd)$.

Full written solutions: AoPS wiki, 2023 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Algebra · A3

Proposed by Merlijn Staps, Netherlands

$x_1, x_2, \ldots, x_{2023}$ are pairwise different positive real numbers such that $a_n = \sqrt{((x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n))}$ is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geq 3034$.

PROOF PROBLEM

KEY IDEA

$a_1 = 1$, and by the Cauchy-Schwarz inequality $a_{n+1} > a_n$, so each step adds at least 1. Two steps in a row cannot both add exactly 1: equality would force $x_{n+1} = x_{n+2}$. So $a_{n+2} \geq a_n + 3$. Going from a_1 to a_{2023} takes 1011 double steps, and $1 + 3 \times 1011 = 3034$.

Full written solutions: AoPS wiki, 2023 IMO Problem 4

5 Problem 5

Combinatorics · C3

Proposed by Merlijn Staps and Daniël Kroes, Netherlands

A Japanese triangle has 1 + 2 + … + n circles in a triangular shape; row i has i circles and exactly one circle in each row is red. A ninja path starts at the top circle and moves, row by row, to one of the two circles just below, ending in the bottom row. Find, in terms of n, the largest k such that every Japanese triangle has a ninja path through at least k red circles.

ANSWER $k = \lfloor \log_2 n \rfloor + 1$.

KEY IDEA

For the upper bound, place the red circles so that a path can pick up at most one red circle from row 1, from rows 2 to 3, from rows 4 to 7, and so on. For the lower bound, give each circle the largest number of red circles on a path that ends there. Tracking how the total of these values in a row grows shows that a value of at least $\lfloor \log_2 n \rfloor + 1$ must appear by row n.

Full written solutions: AoPS wiki, 2023 IMO Problem 5

6 Problem 6

Geometry · G8

Proposed by Ankan Bhattacharya and Luke Robitaille, USA

ABC is an equilateral triangle. Points A_1, B_1, C_1 lie inside it with $BA_1 = A_1C$, $CB_1 = B_1A$, $AC_1 = C_1B$ and $\angle BA_1C + \angle CB_1A + \angle AC_1B = 480°$. Lines BC_1 and CB_1 meet at A_2, lines CA_1 and AC_1 meet at B_2, and lines AB_1 and BA_1 meet at C_2. Prove that if triangle $A_1B_1C_1$ is scalene, the circumcircles of triangles AA_1A_2, BB_1B_2 and CC_1C_2 all pass through two common points.

PROOF PROBLEM

KEY IDEA

Let α, β, γ be the base angles A_1BC, B_1CA, C_1AB; the 480° condition says α + β + γ = 30°. This makes A_1 the circumcentre of triangle A_2BC (and similarly for B_1 and C_1), and from that B_1, C_1, B_2, C_2 lie on a circle γ_a; in the same way C_1, A_1, C_2, A_2 lie on a circle γ_b and A_1, B_1, A_2, B_2 on a circle γ_c. The circle AA_1A_2 is the set of points whose powers with respect to γ_b and γ_c are in a fixed ratio k_a: A_1 and A_2 lie on both circles, and A has that ratio. The same holds for the other two circles with ratios k_b and k_c. A sine-rule computation gives $k_ak_bk_c = 1$, so a point on two of the three circles is also on the third. Two of the circles do cross at two points, so all three pass through both.

Full written solutions: AoPS wiki, 2023 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2023 paper from imo-official.org. Full solutions: AoPS wiki, 2023 IMO and Evan Chen's IMO 2023 solution notes.