IMO 2024 Problems with Solution Outlines

Part 1 · The paper at a glance

65th IMO | Bath, United Kingdom | July 2024

CHART Six problems, two days, 4 hours 30 minutes per day, 7 points per problem (42 in total).

P1	P2	P3	P4	P5	P6
Algebra	Number theory	Combinatorics	Geometry	Combinatorics	Algebra
A1 · Find the answer	N4 · Find the answer	C7 · Proof	G2 · Proof	C4 · Find the answer	A7 · Find the answer
Day 1	Day 1	Day 1	Day 2	Day 2	Day 2

A Day 1 · Problems 1 to 3

1 Problem 1

Algebra · A1

Proposed by Santiago Rodríguez, Colombia

Find all real numbers α such that, for every positive integer n, the integer $\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \ldots + \lfloor n\alpha \rfloor$ is a multiple of n. ($\lfloor z \rfloor$ is the greatest integer not exceeding z.)

ANSWER **Exactly the even integers.**

KEY IDEA

Even integers work because the sum is $\alpha \cdot n(n+1)/2$. Adding 2 to α adds $n(n+1)$ to the sum, so we may assume $0 \le \alpha < 2$. If $0 < \alpha < 1$, take the first n with $n\alpha \ge 1$: the sum is 1 and $n \ge 2$. If $1 \le \alpha < 2$, write $\alpha = 2 - \beta$ and apply a similar first-time argument to β; $\alpha = 1$ fails at $n = 2$. Only $\alpha = 0$ survives.

Full written solutions: AoPS wiki, 2024 IMO Problem 1

2 Problem 2

Number theory · N4

Proposed by Valentio Iverson, Indonesia

Find all pairs (a, b) of positive integers for which there are positive integers g and N such that $\gcd(a^n + b, b^n + a) = g$ for all integers $n \ge N$.

ANSWER **$(a, b) = (1, 1)$.**

KEY IDEA

The number $ab + 1$ is coprime to a and to b. Take large n with $n \equiv -1$ modulo $\varphi(ab + 1)$: then $a^n + b$ and $b^n + a$ are both divisible by $ab + 1$, so $ab + 1$ divides g. Now take large n that is a multiple of $\varphi(ab + 1)$: the two numbers are congruent to $1 + b$ and $1 + a$. So $ab + 1$ divides both $a + 1$ and $b + 1$, which forces $a = b = 1$.

Full written solutions: AoPS wiki, 2024 IMO Problem 2

3 Problem 3

Combinatorics · C7

Proposed by William Steinberg, Australia

$a_1, a_2, a_3, \ldots$ is an infinite sequence of positive integers and N is a positive integer. For each $n > N$, a_n equals the number of times a_{n-1} appears in the list $a_1, a_2, \ldots, a_{n-1}$. Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic.

PROOF PROBLEM

KEY IDEA

For each value v keep a count of how often v has appeared so far. Only finitely many values, say 1, 2, ..., L, appear infinitely often, and from some point on the terms alternate between a value at most L and a value larger than every starting term. Each small term is then determined by the previous small term together with the counts of the values 1 to L, and in fact only by the differences between neighbouring counts. These differences stay bounded, because a count can neither run far ahead of the next one nor fall far behind it. So only finitely many states are possible, a state repeats, and from then on the small terms, which form one of the two subsequences, are periodic.

Full written solutions: AoPS wiki, 2024 IMO Problem 3

B Day 2 · Problems 4 to 6

4 Problem 4

Geometry · G2

Proposed by Dominik Burek, Poland

ABC is a triangle with AB < AC < BC, incentre I and incircle ω. X is the point on line BC, different from C, such that the line through X parallel to AC is tangent to ω. Y is the point on line BC, different from B, such that the line through Y parallel to AB is tangent to ω. Line AI meets the circumcircle of ABC again at P. K and L are the midpoints of AC and AB. Prove that $\angle KIL + \angle YPX = 180°$.

PROOF PROBLEM

KEY IDEA

The two tangent lines are the reflections of AC and AB in I, so they meet at A′, the reflection of A in I. The enlargement with centre A and factor 2 sends K, I, L to C, A′, B, so $\angle KIL = \angle CA'B$. P is the centre of the circle through B, I, C, and an angle chase shows B, P, A′, X are concyclic and C, P, A′, Y are concyclic. Then $\angle YPX = \angle A'BC + \angle A'CB = 180° - \angle CA'B$.

Full written solutions: AoPS wiki, 2024 IMO Problem 4

5 Problem 5

Combinatorics · C4

Proposed by Cheuk Hei Chu, Hong Kong

Turbo the snail plays on a board with 2024 rows and 2023 columns. Monsters hide in 2022 cells: exactly one in each row except the first and last, and at most one in each column. On each attempt Turbo starts anywhere in the first row and moves between cells sharing a side. If he reaches a monster the attempt ends and he returns to the first row; he remembers what he has learned. If he reaches the last row he wins. Find the smallest n for which Turbo has a strategy that is sure to reach the last row by the n-th attempt.

ANSWER **n = 3.**

KEY IDEA

Two attempts are not enough: the first monster Turbo can be made to meet is in row 2, and the second attempt can be blocked just below. For three attempts, Turbo first sweeps row 2 to find its monster. If that monster is not in an end column, Turbo has two paths: go down beside the monster on the left (or on the right), step into the cell directly below the monster, and continue straight down that column, which is now known to be safe. The single monster in row 3 cannot block both. If the row 2 monster is in an end column, Turbo follows a zigzag staircase; if a monster stops him, what he has learned gives a safe path for the third attempt.

Full written solutions: AoPS wiki, 2024 IMO Problem 5

6 Problem 6

Algebra · A7

Proposed by Japan

A function f from the rational numbers to the rational numbers is called aquaesulian if for all rational x and y, $f(x + f(y)) = f(x) + y$ or $f(f(x) + y) = x + f(y)$. Prove that there is an integer c such that, for every aquaesulian f, the expression $f(r) + f(-r)$ takes at most c different values as r runs over the rationals, and find the smallest such c.

ANSWER **c = 2.**

KEY IDEA

First show f is a bijection with $f(0) = 0$ and $f(-f(x)) = -x$. Using the defining property on well-chosen pairs shows that $f(r) + f(-r)$ is 0 or one other fixed value, so at most two values occur. The function $f(x) = \lfloor x \rfloor - \{x\}$, where $\{x\}$ is the fractional part, is aquaesulian and gives two different values, so $c = 2$ cannot be lowered.

Full written solutions: AoPS wiki, 2024 IMO Problem 6

How to use this sheet

Try each problem before you read the key idea. The key idea is a sketch of one route to the solution, not a full proof: use it as a hint, then write the complete argument yourself and compare it with a full written solution.

The problems on this sheet are restated in Gonit's own words. For the official wording, download the IMO 2024 paper from imo-official.org. Full solutions: AoPS wiki, 2024 IMO and Evan Chen's IMO 2024 solution notes.